数理解析研究所講究録
第 1892 巻 2014 年 101-116

Abstract approach to Dirac equation with time dependent potential

東京理科大学・理学部 吉井 健太郎 (Kentarou Yoshii)
Department of Mathematics, Science University of Tokyo

1. Introduction and statement of the result

Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space and

$$H_1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3); (1+|x|^2)^{1/2}u \in L^2(\mathbb{R}^3)\}, \quad \Sigma^1(\mathbb{R}^3) := H^1(\mathbb{R}^3) \cap H_1(\mathbb{R}^3).$$

In this paper we consider the Cauchy problem for Dirac equation in $L^2(\mathbb{R}^3)^4$:

$$\text{(DE)} \qquad i\frac{\partial u}{\partial t} = H_0 u + V(t,x)u, \quad H_0 := \alpha \cdot D + m\beta$$

with $u(\cdot, 0) = u_0 \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$. Here $u : [0,T] \times \mathbb{R}^3 \to \mathbb{C}^4$ is an unknown function and $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, $\beta := \alpha_4$ are 4×4 Hermitian matrices satisfying

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I,$$

where δ_{jk} is Kronecker symbol, I is the identity matrix and $D = i^{-1}(\partial_1, \partial_2, \partial_3)$, $m \geq 0$. $V(t,x)$ is a 4×4 Hermitian matrix-valued potential.

We shall show the existence of a unique classical solution under some conditions on potential V. It seems that this problem is associated with Kato and Yajima [**5**].

Now we want to state our main theorem.

Theorem 1.1. *Set $V = V_0(t,x) + q(t,x)I$, where Hermitian matrix-valued function V_0 and real valued function q satisfy the conditions:*

$$\text{(V1)} \qquad |V_0(t,x)| \leq a|x|^{-1} + b,$$

$$\text{(V2)} \qquad \left|\frac{\partial}{\partial t} V_0(t,x)\right| \leq (a|x|^{-1} + b)\sigma_1(t),$$

$$\text{(q)} \qquad (1+|x|^2)^{-1/2} q \in C(0,T; L^\infty(\mathbb{R}^3)) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}^3)),$$

where $|V|$ denotes the operator norm of V, a, b are nonnegative constants with $a < 1/2$ and $\sigma \in L^1(0,T)$. Then for every initial value $u_0 \in \Sigma^1(\mathbb{R}^3)^4$ problem (DE) *has a unique* (*classical*) *solution*

$$u(\cdot) \in C^1([0,T]; L^2(\mathbb{R}^3)^4) \cap C([0,T]; \Sigma^1(\mathbb{R}^3)^4).$$

Remark 1. Conditions (V1) and (V2) imply $(a|x|^{-1}+b)^{-1}V_0 \in W^{1,1}(0,T;L^\infty(\mathbb{R}^3)^{4\times 4})$.

To prove Theorem 1.1 we employ an abstract approach. This approach becomes a little simple than [**10**].

Let $\{A(t); t \in [0,T]\}$ be a family of closed linear operators in a complex Hilbert space X. Then we consider the abstract Cauchy problem for linear evolution equations of the form

$$\text{(ACP)} \qquad \begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t), & t \in [0,T], \\ u(0) = u_0. \end{cases}$$

Here the initial value u_0 is selected as follows.

To introduce our assumption on $\{A(t);\ t \in [0,T]\}$ we need one more family $\{S(t);\ t \in [0,T]\}$ of auxiliary operators in X.

Assumption on $\{S(t)\}$. The family $\{S(t)\}$ satisfies the following three conditions:

(**S1**) For every $t \in [0,T]$, $S(t)$ is positive selfadjoint in X and

$$(u, S(t)u) \geq \|u\|^2 \quad \text{for } u \in D(S(t)).$$

Let Y_t be the Hilbert space $D(S(t)^{1/2})$ with new inner product $(\cdot\,,\cdot)_{Y_t}$ and norm $\|\cdot\|_{Y_t}$ for $t \in [0,T]$ and $u, v \in Y_t$:

$$(u,v)_{Y_t} := (S(t)^{1/2}u, S(t)^{1/2}v), \qquad \|u\|_{Y_t} := (u,u)_{Y_t}^{1/2},$$

embedded continuously and densely in X. In particular, $Y := Y_0$ plays the roll of the space of initial values as in Theorem 1.3 (see below).

(**S2**) For $t \in [0,T]$, $Y_t = Y$, and $S(\cdot)^{1/2} \in C_*([0,T]; B(Y,X))$, where $B(Y,X)$ is the space of all bounded linear operators on Y to X, with norm $\|\cdot\|_{B(Y,X)}$, while the subscript $*$ is used to refer the strong operator topology in $B(Y,X)$ (for this notation see Kato [**4**]).

(**S3**) There exists a nonnegative function $\sigma \in L^1(I)$ such that

$$\big|\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\|\big| \leq \left|\int_s^t \sigma(r)\,dr\right| \max_{r\in\{s,t\}} \|S(r)^{1/2}v\|, \quad v \in Y,\ t,s \in [0,T].$$

In connection with the symbol $B(Y,X)$ we shall also use the abbreviation: $B(X) := B(X,X)$, $B(Y) := B(Y,Y)$.

Let $\{S(t)\}$ be as defined above. Then we may introduce the following

Assumption on $\{A(t)\}$. The family $\{A(t)\}$ satisfies the following four conditions:

(**A1**) There exists a constant $\alpha \geq 0$ such that

$$|\operatorname{Re}\,(A(t)v,v)| \leq \alpha\|v\|^2, \quad v \in D(A(t)),\ t \in [0,T].$$

(**A2**) $Y \subset D(A(t))$, $t \in [0,T]$.

(**A3**) There exists a constant $\beta \geq \alpha$ such that

$$|\operatorname{Re}\,(A(t)u, S(t)u)| \leq \beta \|S(t)^{1/2}u\|^2, \quad u \in D(S(t)) \subset Y,\ t \in [0,T].$$

(**A4**) $A(\cdot) \in C_*([0,T]; B(Y,X))$.

Theorem 1.2. *Suppose that* Assumptions on $\{A(t)\}$ *and* $\{S(t)\}$ *are satisfied. Then there exists a unique evolution operator* $\{U(t,s); (t,s) \in \Delta_+\}$ *for* (ACP), *where* $\Delta_+ := \{(t,s); 0 \leq s \leq t \leq T\}$, *having the following properties:*

(**i**) $U(\cdot,\cdot)$ *is strongly continuous on* Δ_+ *to* $B(X)$, *with*

$$\|U(t,s)\|_{B(X)} \leq e^{\alpha(t-s)}, \quad (t,s) \in \Delta_+.$$

(**ii**) $U(t,r)U(r,s) = U(t,s)$ *on* Δ_+ *and* $U(s,s) = 1$ (*the identity*).

(**iii**) $U(t,s)Y \subset Y$ *and* $U(\cdot,\cdot)$ *is strongly continuous on* Δ_+ *to* $B(Y)$, *with*

$$\|U(t,s)\|_{B(Y_s,Y_t)} \leq \exp\Big(\int_s^t \widetilde{\sigma}(r)\,dr\Big), \quad (t,s) \in \Delta_+, \tag{1.1}$$

$$\|U(t,s)\|_{B(Y)} \leq \exp\Big(\int_0^s \sigma(r)\,dr\Big) \exp\Big(2\int_s^t \widetilde{\sigma}(r)\,dr\Big), \quad (t,s) \in \Delta_+, \tag{1.2}$$

where $\widetilde{\sigma}(t) := \beta + \sigma(t)$.

Furthermore, let $v \in Y$. *Then* $U(\cdot,\cdot)v \in C^1(\Delta_+; X)$, *with*

(**iv**) $(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v, \quad (t,s) \in \Delta_+$, *and*

(**v**) $(\partial/\partial s)U(t,s)v = U(t,s)A(s)v, \quad (t,s) \in \Delta_+$.

The equation in (ACP) is naturally interpreted if the solution has an additional property $u(\cdot) \in C([0,T]; Y)$. In fact, it is guaranteed by condition (**A2**) that $u(t) \in Y \subset D(A(t))$ for every $t \in [0,T]$.

Theorem 1.3. *Let* $\{U(t,s)\}$ *be the evolution operator for* (ACP) *as in* Theorem 1.2 *above. For* $u_0 \in Y$ *and* $f(\cdot) \in C([0,T]; X) \cap L^1(0,T;Y)$ *define* $u(\cdot)$ *as*

$$u(t) := U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds.$$

Then (ACP) *has a unique* (*classical*) *solution*

$$u(\cdot) \in C^1([0,T]; X) \cap C([0,T]; Y).$$

Remark 2. The assertion of Theorem 1.3 is same as in [**10**]. However, we can simplify assumption on $\{S(t)\}$ and the proof of Theorem 1.2 a little than [**10**].

2. Preliminaries

Let X be a (complex) Hilbert space. In this section we prepare some useful lemmas.

2.1. Lemmas on time-independent operators

In this subsection we consider a pair $\{A, S\}$ of closed linear operators in X. Let A be *quasi-accretive* in the sense of Kato [**1**, Section V.3.10]:

$$\operatorname{Re}(Av, v) \geq -\alpha \|v\|^2, \quad v \in D(A), \tag{2.1}$$

for some constant $\alpha \geq 0$; in other words, $\alpha + A$ is accretive. Let S be a positive-definite selfadjoint operator in X, with $D(S) \subset D(A)$. Now we can state a condition connecting A and S: assume that there exist a constant $\beta \geq 0$ such that

$$\operatorname{Re}(Au, Su) \geq -\beta\,(u, Su), \quad u \in D(S) \subset D(A). \tag{2.2}$$

Lemma 2.1. *Let A and S be as in* (2.1) *and* (2.2), *respectively. Then*

(a) $\alpha + A$ *is m-accretive in* X; (b) $D(S)$ *is a core for* A.

(a) was first proved by Okazawa [**6**], while (b) was later noted by Kato [**3**] (for a complete proof see Tanabe [**13**, Section 7.7] or Ouhabaz [**11**, Section 1.3.3]).

Given the pair $\{A, S\}$ as in Lemma 2.1, let $\{A_n; n > \alpha\}$ and $\{S_\varepsilon; \varepsilon > 0\}$ be Yosida approximations of A and S, respectively:

$$A_n := AJ_n = n(1 - J_n), \qquad J_n := (1 + n^{-1}A)^{-1}, \tag{2.3}$$

$$S_\varepsilon := SJ_\varepsilon = \varepsilon^{-1}(1 - J_\varepsilon), \quad J_\varepsilon := (1 + \varepsilon S)^{-1}. \tag{2.4}$$

Then the pair $\{A_n, S_\varepsilon\}$ satisfies conditions in Lemma 2.1 with α and β replaced with $\alpha(1 - n^{-1}\alpha)^{-1}$ and $\beta(1 - n^{-1}\beta)^{-1}$, respectively.

(2.1) and (2.2) are invariant under taking their Yosida approximation.

Lemma 2.2 (Okazawa [**7**, Lemma 2.2]). *Given A as in* Lemma 2.1, *let $\{A_n\}$ and $\{J_n\}$ be as in* (2.3). *Then $\|J_n\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$ $(n > \alpha)$ and $\|A_n\|_{B(X)} \leq n$ $(n \geq 2\alpha)$, with*

$$\operatorname{Re}\,(A_n w, w) \geq -\alpha(1 - n^{-1}\alpha)^{-1}\|w\|^2, \quad w \in X, \quad n > \alpha. \tag*{$(2.1)_n$}$$

Lemma 2.3 ([**9**, Lemmas 2.7]). *Given the pair $\{A, S\}$ as in* Lemma 2.1, *let $\{A_n\}$ and $\{S_\varepsilon\}$ be as in* (2.3) *and* (2.4). *Then*

$$\operatorname{Re}\,(A_n w, S_\varepsilon w) \geq -\beta(1 - n^{-1}\beta)^{-1}\|S_\varepsilon^{1/2} w\|^2, \quad w \in X, \quad n > \beta \geq \alpha. \tag*{$(2.2)_{n,\varepsilon}$}$$

2.2. Lemmas on time-dependent operators

Since we need conditions (**A1**), (**A3**) and (**S3**) as a whole only in the last step of the proof of Theorem 1.2, we may introduce a set of weaker conditions:

(**A1**)$_+$ There exists a constant $\alpha \geq 0$ such that

$$\mathrm{Re}\,(A(t)v, v) \geq -\alpha\|v\|^2, \quad v \in D(A(t)),\ t \in [0,T].$$

(**A3**)$_+$ There exists a constant $\beta \geq \alpha$ such that

$$\mathrm{Re}\,(A(t)v, S(t)v) \geq -\beta\|S(t)^{1/2}v\|^2, \quad v \in D(S(t)) \subset Y,\ t \in [0,T].$$

(**S3**)$_+$ There exists a nonnegative function $\sigma \in L^1(0,T)$ such that

$$\|S(t)^{1/2}u\| - \|S(s)^{1/2}u\| \leq \Big(\int_s^t \sigma(r)\,dr\Big)\|S(s)^{1/2}u\|, \quad u \in Y,\ 0 \leq s \leq t \leq T.$$

By virtue of Lemma 2.1, it follows from conditions (**A1**)$_+$, (**A2**) and (**A3**)$_+$ that $\alpha + A(t)$ is m-accretive in X. We can obtain following

Proposition 2.4. *Let $\{S(t)\}$ be a family of selfadjoint operators in X, $\{A(t)\}$ a family of closed linear quasi-accretive operators in X, satisfying conditions* (**S1**), (**S2**), (**S3**)$_+$ *for $\{S(t)\}$, and conditions* (**A1**)$_+$, (**A2**), (**A3**)$_+$, (**A4**) *for $\{A(t)\}$. Then there exist Yosida approximations $\{A_n(t); n > \alpha\}$ and $\{S_\varepsilon(t); \varepsilon > 0\}$ of $\{A(t)\}$ and $\{S(t)\}$, respectively*:

$$A_n(t) := A(t)J_n(t) = n(1 - J_n(t)), \qquad J_n(t) := (1 + n^{-1}A(t))^{-1}, \tag{2.5}$$

$$S_\varepsilon(t) := S(t)J_\varepsilon(t) = \varepsilon^{-1}(1 - J_\varepsilon(t)), \qquad J_\varepsilon(t) := (1 + \varepsilon S(t))^{-1}. \tag{2.6}$$

Put

$$\alpha_n := \alpha(1 - n^{-1}\alpha)^{-1}, \qquad \beta_n := \beta(1 - n^{-1}\beta)^{-1} \quad (n > \beta \geq \alpha). \tag{2.7}$$

Then the pair of $\{A_n(t)\}$ and $\{S_\varepsilon(t)\}$ satisfies Assumptions *on $\{A(t)\}$ and $\{S(t)\}$ with Y, α and β replaced with X, α_n and β_n, respectively*:

(**S$_\varepsilon$1**) $D(S_\varepsilon(t)) = X$, $t \in [0,T]$.

(**S$_\varepsilon$2**) $S_\varepsilon(\cdot) \in C_*([0,T]; B(X))$.

(**S$_\varepsilon$3**)$_+$ *For $w \in X$,*

$$\|S_\varepsilon(t)^{1/2}w\|^2 - \|S_\varepsilon(s)^{1/2}w\|^2 \leq \Big[\Big(1 + \int_s^t \sigma(r)\,dr\Big)^2 - 1\Big]\|S_\varepsilon(s)^{1/2}w\|^2, \quad 0 \leq s \leq t \leq T.$$

(**A$_n$1**)$_+$ *For $n > \alpha$, $t \in [0,T]$, $\|J_n(t)\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$ and*

$$\mathrm{Re}\,(A_n(t)w, w) \geq -\alpha_n\|w\|^2, \quad w \in X.$$

(**A$_n$2**) $D(A_n(t)) = X$, $t \in [0,T]$.

(**A$_n$3**)$_+$ *For $n > \beta$, $t \in [0,T]$,* $\mathrm{Re}\,(A_n(t)w, S_\varepsilon(t)w) \geq -\beta_n\|S_\varepsilon(t)^{1/2}w\|^2, \quad w \in X$.

(**A$_n$4**) $A_n(\cdot) \in C_*([0,T]; B(X))$ $(n > \beta)$, *with* $\|A_n(t)\|_{B(X)} \leq n$ $(n \geq 2\alpha,\ t \in [0,T])$.

($\mathbf{S}_\varepsilon\mathbf{1}$) and ($\mathbf{A}_n\mathbf{2}$) are trivial. $(\mathbf{A}_n\mathbf{1})_+$ and $(\mathbf{A}_n\mathbf{3})_+$ follow from Lemmas 2.2 and 2.3, respectively. ($\mathbf{A}_n\mathbf{4}$) and ($\mathbf{S}_\varepsilon\mathbf{2}$) were proved in [**7**, Lemma 3.1 (a), (b)] and [**10**, Proposition 2.5], respectively. We show only the proof of $(\mathbf{S}_\varepsilon\mathbf{3})_+$. The first half of the proof is same as in [**10**, Proposition 2.5].

Proof of $(\mathbf{S}_\varepsilon\mathbf{3})_+$. First we remind of the definition of $S_\varepsilon(\cdot)$:

$$w = J_\varepsilon(t)w + \varepsilon S_\varepsilon(t)w, \quad w \in X.$$

Then the symmetry and positivity of $J_\varepsilon(\cdot)$ yield that

$$\begin{aligned}
(2.8)\qquad & (w, S_\varepsilon(t)w - S_\varepsilon(s)w) \\
&= \big(w, S_\varepsilon(t)(J_\varepsilon(s) + \varepsilon S_\varepsilon(s))w - (J_\varepsilon(t) + \varepsilon S_\varepsilon(t))S_\varepsilon(s)w\big) \\
&= \big(J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big) + \big((J_\varepsilon(t) - J_\varepsilon(s))w, (S(t) - S(s))J_\varepsilon(s)w\big) \\
&= \big(J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big) \\
&\quad - \varepsilon\big(J_\varepsilon(t)(S(t) - S(s))J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big) \\
&\le \big(J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w\big) \\
&= \|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2.
\end{aligned}$$

On the other hand, it follows from condition $(\mathbf{S3})_+$ that

$$\|S(t)^{1/2}J_\varepsilon(s)w\| \le \Big(1 + \int_s^t \sigma(r)\,dr\Big)\|S_\varepsilon(s)^{1/2}w\|, \tag{2.9}$$

where we have used $\|J_\varepsilon(s)^{1/2}\|_{B(X)} \le 1$ $(\varepsilon > 0)$. We see from (2.9) that

$$\|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2 \le \Big[\Big(1 + \int_s^t \sigma(r)\,dr\Big)^2 - 1\Big]\|S_\varepsilon(s)^{1/2}w\|^2. \tag{2.10}$$

Combining (2.10) with (2.8), we obtain $(\mathbf{S}_\varepsilon\mathbf{3})_+$. □

The next proposition is essential in the proof of Theorem 1.2 in which it is required to differentiate $S_\varepsilon(\cdot)$. For that purpose $\{S_\varepsilon(t);\ t \in [0,T]\}$ is replaced with a new family

$$S_\varepsilon^h(t) := \frac{1}{h}\int_t^{t+h} S_\varepsilon(s)\,ds, \quad h > 0,\ t \in [0,T], \tag{2.11}$$

where we define as $S_\varepsilon(s) := S_\varepsilon(T)$ $(s > T)$. Then, in view of ($\mathbf{S}_\varepsilon\mathbf{2}$), we have

$$\frac{d}{dt}S_\varepsilon^h(t)w = \frac{1}{h}(S_\varepsilon(t+h)w - S_\varepsilon(t)w), \tag{2.12}$$

$$\operatorname*{s-lim}_{h\downarrow 0} S_\varepsilon^h(t)w = S_\varepsilon(t)w, \quad w \in X. \tag{2.13}$$

Proposition 2.5. *Assume that $\{S(t)\}$ satisfies conditions* (**S1**), (**S2**) *and* $(\mathbf{S3})_+$. *Let* $\{S_\varepsilon^h(t);\, t \in [0,T]\}$ *be as in* (2.11). *Then* $S_\varepsilon^h(\cdot) \in C_*^1([0,T]; B(X))$, *with for* $w \in X$,

$$(\mathbf{S}_\varepsilon^h\mathbf{3})_+ \qquad \Big(w, \frac{d}{ds}S_\varepsilon^h(s)w\Big) \le \frac{1}{h}\Big[\Big(1+\int_s^{s+h}\sigma(r)\,dr\Big)^2 - 1\Big]\|S_\varepsilon(s)^{1/2}w\|^2, \qquad s \in [0,T],$$

where one defines as $\sigma(s) := 0$ $(s > T)$. *Moreover, for* $w \in X$ *one has*

$$\limsup_{h\downarrow 0}\int_{t_0}^t \Big(w, \frac{d}{ds}S_\varepsilon^h(s)w\Big)\,ds \le 2\int_{t_0}^t \sigma(s)\|S_\varepsilon(s)^{1/2}w\|^2\,ds, \quad t_0 \le t. \tag{2.14}$$

With regard to the proof of Proposition 2.6, $(\mathbf{S}_\varepsilon^h\mathbf{3})_+$ follows from (2.12) and $(\mathbf{S}_\varepsilon\mathbf{3})_+$. On the other hand, (2.14) is a consequence of following:

Lemma 2.6. *Let* $\varphi \in L^1(0,T)$ *and* $\psi \in L^\infty(0,T)$. *Then*

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^t\Big[\Big(1+\int_s^{s+h}\varphi(r)\,dr\Big)^2 - 1\Big]\psi(s)\,ds = 2\int_0^t \varphi(s)\psi(s)\,ds, \quad 0 \le t \le T,$$

where one define as $\varphi(s) := 0$ $(s > T)$.

Proof. We also define that $\varphi(s) := 0$ $(s < 0)$. Since

$$\Big(1+\int_s^{s+h}\varphi(r)\,dr\Big)^2 - 1 = \Big(\int_s^{s+h}\varphi(r)\,dr\Big)\Big(2+\int_s^{s+h}\varphi(\tau)\,d\tau\Big),$$

by integrating by parts, we obtain

$$\begin{aligned}
&\int_0^t\Big[\Big(1+\int_s^{s+h}\varphi(r)\,dr\Big)^2 - 1\Big]\psi(s)\,ds \\
&= \int_0^{t+h}\varphi(r)\Big[\int_{r-h}^r\Big(2+\int_s^{s+h}\varphi(\tau)\,d\tau\Big)\widetilde{\psi}(s)\,ds\Big]\,dr \\
&= \int_0^{t+h}\varphi(r)\Big[\int_{r-h}^{r+h}\Big(2\overline{\psi}(s)+\varphi(\tau)\int_{\tau-h}^{\tau}\overline{\psi}(s)\,ds\Big)\,d\tau\Big]\,dr,
\end{aligned}$$

where $\widetilde{\psi}$, $\overline{\psi} \in L^1(\mathbb{R})$ are defined as

$$\widetilde{\psi}(s) := \begin{cases}\psi(s), & 0 < s < t,\\ 0, & \text{otherwise},\end{cases} \qquad \overline{\psi}(s) := \begin{cases}\widetilde{\psi}(s), & r-h < s < r,\\ 0, & \text{otherwise}.\end{cases}$$

Set

$$\Phi_h(r) := \frac{1}{h}\varphi(r)\Big[\int_{r-h}^{r+h}\Big(2\overline{\psi}(s)+\varphi(\tau)\int_{\tau-h}^{\tau}\overline{\psi}(s)\,ds\Big)\,d\tau\Big].$$

Then we obtain $\lim_{h\downarrow 0}\Phi_h(r) = 2\varphi(r)\widetilde{\psi}(r)$ (a.a. $r \in (0,t)$) and

$$\begin{aligned}
|\Phi_h(r)| &\le |\varphi(r)|\Big(2+\int_{r-h}^{r+h}|\varphi(\tau)|\,d\tau\Big)\|\psi\|_{L^\infty(0,T)} \\
&\le |\varphi(r)|\Big(2+\|\varphi\|_{L^1(0,T)}\Big)\|\psi\|_{L^\infty(0,T)} \in L^1(0,T).
\end{aligned}$$

Thus the assertion follows from the Lebesgue convergence theorem. □

3. Construction of evolution operators

In this section we shall prove Theorem 1.2. The major part of the assertions in Theorem 1.2 is contained in the following

Theorem 3.1. *Let $\{S(t)\}$ be a family of selfadjoint operators in X, $\{A(t)\}$ a family of closed linear quasi-accretive operators in X, satisfying conditions* (**S1**), (**S2**), $(\mathbf{S3})_+$ *for $\{S(t)\}$, and conditions* $(\mathbf{A1})_+$, (**A2**), $(\mathbf{A3})_+$, (**A4**) *for $\{A(t)\}$. Then there exists a unique two-parameter family $\{U(t,s); (t,s) \in \Delta_+\}$ in $B(X)$, having the properties:*

$(\mathbf{iii})_{\mathrm{w}}$ *$U(t,s)Y \subset Y$, with* (1.1);

$(\mathbf{iv})_{\mathrm{w}}$ *$U(\cdot,\cdot)v \in W^{1,1}(\Delta_+; X)$, with*

$$(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v, \quad \text{a.a. } t \in (s,T), \ v \in Y;$$

in addition to properties (**i**), (**ii**) *and* (**v**) *in* Theorem 1.2.

Therefore the first purpose of this section is to prove Theorem 3.1. To replace $(\mathbf{iii})_{\mathrm{w}}$ and $(\mathbf{iv})_{\mathrm{w}}$ with (**iii**) and (**iv**) in the final step we need the whole conditions (**S1**)–(**S3**) and (**A1**)–(**A4**) in Theorem 1.2.

To prove Theorem 3.1 we consider the approximate problem:

$$(\mathrm{ACP})_n \qquad \begin{cases} (d/dt)u_n(t) + A_n(t)u_n(t) = 0, & t \in [s,T], \\ u_n(s) = w \in X, \end{cases}$$

where $\{A_n(t); n > \alpha\}$ is the Yosida approximation as in (2.5).

According to Pazy [**12**, Theorems 5.1.1 and 5.1.2] (in which $A_n(\cdot) \in C([0,T]; B(X))$ is assumed, however, it can be replaced by $A_n(\cdot) \in C_*([0,T]; B(X))$ (condition $(\mathbf{A}_n\mathbf{4})$) with appropriate modification of the conclusion), we obtain the following

Proposition 3.2. *Let $s \in [0,T)$ and $n > 2\beta$, where β is defined in* $(\mathbf{A3})_+$. *Then the approximate problem* $(\mathrm{ACP})_n$ *has a unique classical solution $u_n(\cdot) \in C^1([s,T]; X)$. Accordingly, there exists a unique evolution operator $\{U_n(t,s); (t,s) \in \Delta_+\}$ for* $(\mathrm{ACP})_n$ *having the following properties:*

$(\mathbf{i})_n$ *$U_n(\cdot,\cdot)$ is strongly continuous on Δ_+ to $B(X)$, with*

$$\|U_n(t,s)\|_{B(X)} \le e^{n(t-s)}, \quad (t,s) \in \Delta_+.$$

$(\mathbf{ii})_n$ *$U_n(t,r)U_n(r,s) = U_n(t,s)$ on Δ_+ and $U_n(s,s) = 1$ (the identity).*

$(\mathbf{iii})_n$ *$U_n(t,s)$ is uniformly continuous on Δ_+.*

$(\mathbf{iv})_n$ *$(\partial/\partial t)U_n(t,s)w = -A_n(t)U_n(t,s)w, \quad w \in X, \ (t,s) \in \Delta_+$.*

$(\mathbf{v})_n$ *$(\partial/\partial s)U_n(t,s)v = U_n(t,s)A_n(s)v, \quad w \in X, \ (t,s) \in \Delta_+$.*

For the limiting procedure for $\{U_n(t,s)\}$ we need several estimates of $\{U_n(t,s)\}$ which are independent of n.

Lemma 3.3. *Let* $\{U_n(t,s)\}$ *be as in* Proposition 3.2, α_n, β_n *as in* (2.7) *and* σ *as in condition* $(\mathbf{S3})_+$. *If* $n > 2\beta$, *then for* $(t,s) \in \Delta_+$,

(a) $\|U_n(t,s)\|_{B(X)} \le e^{\alpha_n(t-s)}$.

(b) $U_n(t,s)Y \subset Y$ *and with*

$$\|S(t)^{1/2}U_n(t,s)v\| \le e^{\beta_n(t-s)} \exp\Big(\int_s^t \sigma(r)\,dr\Big)\|S(s)^{1/2}v\|, \quad v \in Y.$$

(c) *There exists a constant* $c \ge 0$ *such that*

$$\|A(t)v\| \le c\|S(t)^{1/2}v\|, \quad v \in Y, \tag{3.1}$$

and hence

$$\|A_n(t)U_n(t,s)v\| \le M\|v\|_Y, \quad v \in Y, \tag{3.2}$$

where $M := 2c\exp(2\beta T + \|\sigma\|_{L^1(0,T)})$.

Proof. (a) We see from property $(\mathbf{iv})_n$ and $(\mathbf{A}_n\mathbf{1})_+$ that for $w \in X$,

$$\begin{aligned}(\partial/\partial r)\|U_n(r,s)w\|^2 &= -2\,\mathrm{Re}\,(A_n(r)U_n(r,s)w, U_n(r,s)w)\\ &\le 2\alpha_n\|U_n(r,s)w\|^2, \quad s \le r \le t.\end{aligned}$$

Integrating this inequality, we obtain the assertion.

(b) Let $\{S_\varepsilon(t)\}$ and $\{S_\varepsilon^h(t)\}$ be as in (2.6) and (2.11). Since $S_\varepsilon^h(t)^{1/2}$ is bounded and symmetric on X, it follows from property $(\mathbf{iv})_n$ that for $v \in Y$,

$$\begin{aligned}(\partial/\partial r)\|S_\varepsilon^h(r)^{1/2}U_n(r,s)v\|^2 &= -2\,\mathrm{Re}\,\big(A_n(r)U_n(r,s)v, S_\varepsilon^h(r)U_n(r,s)v\big)\\ &\quad + \big(U_n(r,s)v, ((d/dr)S_\varepsilon^h(r))U_n(r,s)v\big).\end{aligned} \tag{3.3}$$

Integrating this equality on $[s,t]$, we see from $(\mathbf{S}_\varepsilon^h\mathbf{3})_+$ that

$$\begin{aligned}\|S_\varepsilon^h(t)^{1/2}U_n(t,s)v\|^2 &\le \|S_\varepsilon^h(s)^{1/2}v\|^2 - 2\int_s^t \mathrm{Re}\,\big(A_n(r)U_n(r,s)v, S_\varepsilon^h(r)U_n(r,s)v\big)\,dr\\ &\quad + \int_s^t \frac{1}{h}\Big[\Big(1 + \int_r^{r+h}\sigma(\tau)\,d\tau\Big)^2 - 1\Big]\|S_\varepsilon(r)^{1/2}U_n(r,s)v\|^2\,dr.\end{aligned}$$

Passing to the limit as $h \downarrow 0$, we see from (2.13), (2.14) and $(\mathbf{A}_n\mathbf{3})_+$ that

$$\begin{aligned}\|S_\varepsilon(t)^{1/2}U_n(t,s)v\|^2 &\le \|S_\varepsilon(s)^{1/2}v\|^2 - 2\int_s^t \mathrm{Re}\,\big(A_n(r)U_n(r,s)v, S_\varepsilon(r)U_n(r,s)v\big)\,dr\\ &\quad + 2\int_s^t \sigma(r)\|S_\varepsilon(r)^{1/2}U_n(r,s)v\|^2\,dr\\ &\le \|S(s)^{1/2}v\|^2 + 2\int_s^t (\beta_n + K\sigma(r))\,\|S_\varepsilon(r)^{1/2}U_n(r,s)v\|^2\,dr.\end{aligned}$$

Applying the Gronwall lemma, we obtain

$$\|S_\varepsilon(t)^{1/2}U_n(t,s)v\|^2 \le \exp\Big(2\int_s^t (\beta_n + \sigma(r))\,dr\Big)\|S(s)^{1/2}v\|^2.$$

Passing to the limit as $\varepsilon \to 0$, we obtain the assertion.

(c) The existence of such a constant c is guaranteed by conditions (**A2**) and (**A4**). Therefore we see from $(\mathbf{A}_n\mathbf{1})_+$ and (3.1) that

$$\|A_n(t)v\| \le (1 - n^{-1}\alpha)^{-1}\|A(t)v\| \le 2c\|S(t)^{1/2}v\|, \quad v \in Y.$$

Noting that

$$\|S(s)^{1/2}v\| \le \Big(1 + \int_0^s \sigma(r)\,dr\Big)\|S(0)^{1/2}v\| \le \exp\Big(\int_0^s \sigma(r)\,dr\Big)\|v\|_Y, \tag{3.4}$$

we obtain (3.2) as a consequence of (b). □

Next two lemmas guarantee the existence and uniqueness of evolution operator.

Lemma 3.4 ([**10**, Lemma 3.4]). *Let $\{U_n(t,s)\}$ be the evolution operator for* $(\mathrm{ACP})_n$. *Then there exists a new family $\{U(t,s); (t,s) \in \Delta_+\}$ such that $U(t,s) := \operatorname*{s\text{-}lim}_{n\to\infty} U_n(t,s)$, where the convergence is uniform on Δ_+, and has properties* (**i**) *and* (**ii**) *in* Theorem 1.2, *with*

$$\|U(t,s)v - U_n(t,s)v\| \le \sqrt{\frac{2T}{n-2\alpha}}Me^{2\alpha T}\|v\|_Y, \ v \in Y, \ n > 2\beta.$$

Lemma 3.5 ([**10**, Lemma 3.4]). *Let $\{U(t,s)\}$ be as in* Lemma 3.4 *and $v \in Y$. Then*

(a) *$U(t,s)Y \subset Y$ and*

$$\|S(t)^{1/2}U(t,s)v\| \le \exp\Big(\int_s^t \widetilde{\sigma}(r)\,dr\Big)\|S(s)^{1/2}v\|, \quad (t,s) \in \Delta_+, \ v \in Y, \tag{3.5}$$

where $\widetilde{\sigma}$ is defined as in Theorem 1.2 (**iii**).

(b) *$U(\cdot,\cdot)v \in W^{1,\infty}(\Delta_+;X)$, with properties* $(\mathbf{iv})_{\mathrm{w}}$ *and* (**v**).

(c) *$\{U(t,s);\ (t,s) \in \Delta_+\}$ is unique: $U(t,s) \equiv V(t,s)$ on Δ_+ if $\{V(t,s);\ (t,s) \in \Delta_+\}$ is another family in $B(X)$ with properties* (**i**), (**ii**) *and* (**v**).

This completes the proof of Theorem 3.1. The purpose of the second half in this section is to prove properties (**iii**) and (**iv**) in Theorem 1.2.

Lemma 3.6. *Let $\{U(t,s)\}$ be as in* Lemma 3.4 *and $v \in Y$. Assume that $S(t)$ satisfies conditions* (**S1**)–(**S3**). *Then*

(a) *$S(t)^{1/2}U(t,s)v$ is weakly continuous on Δ_+.*

(b) *$S(t)^{1/2}U(t,s)v \to S(t_0)^{1/2}v$ as $(t,s) \to (t_0,t_0)$.*

(c) *For $t \in (0,T]$, $S(t)^{1/2}U(t,\cdot)v \in C([0,t];X)$.*

Proof. (a) Since $S_\varepsilon(\cdot)^{1/2} \in C_*([0,T]; B(X))$ (see $(\mathbf{S}_\varepsilon\mathbf{2})$), $S_\varepsilon(t)^{1/2}U(t,s)v$ is continuous on Δ_+. For $w \in Y$, we see that

$$\begin{aligned}
&\left|(S(t)^{1/2}U(t,s)v, w) - (S_\varepsilon(t)^{1/2}U(t,s)v, w)\right| \\
&= \left|(S(t)^{1/2}U(t,s)v, [1 - (1+\varepsilon S(t))^{-1/2}]w)\right| \\
&\le \exp(\|\widetilde{\sigma}\|_{L^1(0,T)})\|S(0)^{1/2}v\| \cdot \left\|w - (1+\varepsilon S(t))^{-1/2}w\right\| \\
&\le \varepsilon^{1/2}\exp(\|\widetilde{\sigma}\|_{L^1(0,T)})\|v\|_Y\|w\|_Y,
\end{aligned}$$

where we have used (1.2) and (3.4). Passing to $\varepsilon \downarrow 0$, since Y is dense in X, we can conclude (a).

(b) It is follows from (a) that

$$\|S(t_0)^{1/2}v\| \le \liminf_{(t,s)\to(t_0,t_0)} \|S(t)^{1/2}U(t,s)v\|.$$

On the other hand, it follows from (3.5) that

$$\limsup_{(t,s)\to(t_0,t_0)} \|S(t)^{1/2}U(t,s)v\| \le \|S(t_0)^{1/2}v\|.$$

Combining these estimates and (a), we obtain the assertion.

(c) Let $r \in [s,t]$. Then we see from (3.5) that

$$\|S(t)^{1/2}U(t,r)v - S(t)^{1/2}U(t,s)v\| \le \exp(\|\widetilde{\sigma}\|_{L^1(0,T)})\|S(r)^{1/2}(1-U(r,s))v\|.$$

Therefore the assertion follows from (b). □

Now we are in a position to prove (**iii**) and (**iv**) of Theorem 1.2. As is easily seen, the proof of (**iii**) and (**iv**) is based on Lemmas 3.7 and 3.8 below. In other words, we need the whole assumptions on $\{A(t)\}$ and $\{S(t)\}$.

Lemma 3.7 ([**7**, Lemma 3.9]). *Let* $\{A(t)\}$ *and* $\{S(t)\}$ *be as in* Theorem 1.2. *Assume that conditions* (**A1**), (**A2**) *and* (**A3**) *are satisfied. Then*

$$|\,\mathrm{Re}\,(A(t)v, S_\varepsilon(t)v)| \le \beta\|S_\varepsilon(t)^{1/2}v\|^2, \quad v \in Y, \quad t \in [0,T]. \tag{3.6}$$

Under conditions (**S1**)–(**S3**) Proposition 2.5 is modified as follows.

Lemma 3.8. *Assume that* $\{S(t)\}$ *satisfies conditions* (**S1**)–(**S3**). *Let* $\{S_\varepsilon^h(t)\}$ *be as in* Proposition 2.5. *Then condition* $(\mathbf{S}_\varepsilon^h\mathbf{3})_+$ *is replaced with*

$$\left|\left(w, \frac{d}{ds}S_\varepsilon^h(s)w\right)\right| \le \frac{1}{h}\left[\left(1 + \int_s^{s+h}\sigma(r)\,dr\right)^2 - 1\right]\max_{r\in\{s,s+h\}}\|S_\varepsilon(r)^{1/2}w\|^2 \tag{$\mathbf{S}_\varepsilon^h\mathbf{3}$}$$

for $w \in X$, $h > 0$ *and* $s \in [0,T]$. *Consequently, for* t, $t_0 \in [0,T]$ *one has*

$$\limsup_{h\downarrow 0}\left|\int_{t_0}^t\left|\left(w, \frac{d}{ds}S_\varepsilon^h(s)w\right)\right|ds\right| \le 2\left|\int_{t_0}^t \sigma(s)\|S_\varepsilon(s)^{1/2}w\|^2\,ds\right|. \tag{3.7}$$

The next lemma completes the proof of Theorem 1.2.

Lemma 3.9. *For* $\{A(t)\}$ *and* $\{S(t)\}$ *as in* Theorem 1.2 *let* $\{U(t,s)\}$ *be as defined in* Lemma 3.4 *and* $v \in Y$. *Then*

(a) $U(t,\cdot)v \in C([0,t];Y)$, $t \in (0,T]$.

(b) $S(\cdot)^{1/2}U(\cdot,s)v \in C([s,T];X)$, $s \in [0,T)$.

(c) $U(\cdot,s)v \in C([s,T];Y)$, $s \in [0,T)$.

(d) $U(\cdot,\cdot)v \in C(\Delta_+;Y)$; *this establishes property* **(iii)** *in* Theorem 1.2.

(e) $U(\cdot,\cdot)v \in C^1(\Delta_+;X)$, *with property* **(iv)** *in* Theorem 1.2.

Proof. We follow the idea in [**7**, Lemma 3.10].

(a) We see from condition **(S3)** that for $(t,s) \in \Delta_+$,

$$\|S(s)^{1/2}w\| \le \exp\Big(\int_s^t \sigma(r)\,dr\Big)\|S(t)^{1/2}w\|, \quad w \in Y. \tag{3.8}$$

Hence we obtain

$$\|U(t,r)v - U(t,s)v\|_Y \le \exp(\|\sigma(r)\|_{L^1(0,T)})\|S(t)^{1/2}U(t,r)v - S(t)^{1/2}U(t,s)v\|.$$

Therefore the assertion follows from Lemma 3.6 (c).

(b) By virtue of Lemma 3.6 (a), it suffices to show that

$$\|S(\cdot)^{1/2}U(\cdot,s)v\| \in C[s,T]. \tag{3.9}$$

We trace the proof of Lemma 3.3 (b). Let us t, $t_0 \in [s,T]$. Integrating the inequality (3.3) from $r = t_0$ to $r = t$ and passing to the limit as $n \to \infty$, we have

$$\begin{aligned}
&\|S_\varepsilon^h(t)^{1/2}U(t,s)v\|^2 - \|S_\varepsilon^h(t_0)^{1/2}U(t_0,s)v\|^2\\
&= -2\int_{t_0}^t \mathrm{Re}\,\big(A(r)U(r,s)v, S_\varepsilon^h(r)U(r,s)v\big)\,dr\\
&+ \int_{t_0}^t \Big(U(r,s)v, \Big(\frac{d}{dr}S_\varepsilon^h(r)\Big)U(r,s)v\Big)\,dr.
\end{aligned}$$

Passing to the limit as $h \downarrow 0$, we see from (3.7) that

$$\begin{aligned}
&\big|\|S_\varepsilon(t)^{1/2}U(t,s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0,s)v\|^2\big|\\
&\le 2\Big|\int_{t_0}^t \big|\mathrm{Re}\,\big(A(r)U(r,s)v, S_\varepsilon(r)U(r,s)v\big)\big|\,dr\Big|\\
&+ 2\Big|\int_{t_0}^t \sigma(r)\|S_\varepsilon(r)^{1/2}U(r,s)v\|^2\,dr\Big|.
\end{aligned}$$

Therefore (3.6) yields that

$$\left|\|S_\varepsilon(t)^{1/2}U(t,s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0,s)v\|^2\right| \le 2\left|\int_{t_0}^t \widetilde{\sigma}(r)\|S(r)^{1/2}U(r,s)v\|^2\,dr\right|.$$

By virtue of (3.5) and (3.4), we have

$$\left|\|S_\varepsilon(t)^{1/2}U(t,s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0,s)v\|^2\right| \le 2\exp(2\|\widetilde{\sigma}\|_{L^1(0,T)})\left|\int_{t_0}^t \widetilde{\sigma}(\tau)\,dr\right|\|v\|_Y^2.$$

Passing to the limit as $\varepsilon \downarrow 0$, we obtain (3.9).

(c) We see from (3.8) that

$$\|U(t,s)v - U(t_0,s)v\|_Y \le \exp(\|\widetilde{\sigma}\|_{L^1(0,T)})\|S(t)^{1/2}U(t,s)v - S(t)^{1/2}U(t_0,s)v\|.$$

The assertion is a consequence of (b) and condition (**S2**).

(d) The assertion follows from (c) and Lemma 3.6 (c) (see also [**9**, Lemma 3.11 (b)] and Kato [**2**, Remark 5.4]). Since (3.4), (3.8) and Lemma 3.5 (a) yield (1.2), this completes the proof of property (**iii**).

(e) By virtue of Lemma 3.5 (b), it suffices to show that $A(\cdot)U(\cdot,s)v \in C([s,T];X)$.Let $t,\, t_0 \in [s,T]$. Then we see from (3.1) that the desired continuity is reduced to those of $S_0^{1/2}U(\cdot,s)v$ and $A(\cdot)U(t_0,s)v$:

$$\begin{aligned}
&\|A(t)U(t,s)v - A(t_0)U(t_0,s)v\| \\
&\le \|A(t)U(t,s)v - A(t)U(t_0,s)v\| + \|A(t)U(t_0,s)v - A(t_0)U(t_0,s)v\| \\
&\le c\|S(t)^{1/2}U(t,s)v - S(t)^{1/2}U(t_0,s)v\| + \|A(t)U(t_0,s)v - A(t_0)U(t_0,s)v\|.
\end{aligned}$$

Therefore the conclusion follows from (c) and condition (**A4**). Finally, property (**iv**) is a consequence of property $(\mathbf{iv})_{\mathrm{w}}$. □

4. Applications to the Dirac equation

In this section we consider, as an application of Theorem 1.3, the Cauchy problem for the Dirac equation:

$$\text{(DE)} \qquad \begin{cases} i\dfrac{d}{dt}u = H(t)u & \text{for } t \in (0,T), \\ u(0) = u_0 \end{cases}$$

in the Hilbert Space $X = L^2(\mathbb{R}^3)^4$, where $u_0 \in Y := \Sigma^1(\mathbb{R}^3)^4$.

First we define an operator $H(t)$. Let

$$\mathcal{H}(t) := H_0 + V(t,x) = H_0 + V_0(t,x) + q(t)I$$

with domain $D(\mathcal{H}(t)) = C_0^\infty(\mathbb{R}^3)^4$, where H_0 is the free Dirac operator, $V_0(t,x)$ and $q(t,x)$ are satisfying condition (V1), (V2) and (q). Since $\mathcal{H}(t)$ is symmetric, $\mathcal{H}(t)$ is closable. Then we take as $H(t)$ the closure $\widetilde{\mathcal{H}}(t)$ of $\mathcal{H}(t)$, i.e., $H(t) = \widetilde{\mathcal{H}}(t)$. Set

$$\begin{aligned} S(t) &:= (H_0 + V_0(t))^2 + (1+|x|^2)I, \\ D(S(t)) &:= \{u \in L^2(\mathbb{R}^3)^4; S(t)u \in L^2(\mathbb{R}^3)^4\}. \end{aligned}$$

Then we can show that $S(t)$ is positive selfadjoint on $D(S(t))$ (see [**8**, Lemma 5.4]) and $Y_t := D(S(t)^{1/2})$ is regarded as a Hilbert space, embedded continuously and densely in $L^2(\mathbb{R}^3)^4$, with inner product

$$(u,v)_{Y_t} = (S(t)^{1/2}u, S(t)^{1/2}v), \quad u,\, v \in Y_t.$$

We can show that $S(t)$ satisfies condition (**S1**) and the first half of condition (**S2**):

Lemma 4.1 (cf. [**8**, Lemma 6.1]). *Let $S(t)$ be as above. Then for $t \in [0,T]$,*

$$Y_t = Y = \Sigma^1(\mathbb{R}^3)^4$$

and there exist time independent positive constants c_1, c_2 such that

$$c_1\|S(t)^{1/2}u\|^2 \le \|u\|^2 + \|\nabla u\|^2 + \||x|u\|^2 \le c_2\|S(t)^{1/2}u\|^2, \quad u \in Y. \tag{4.1}$$

Moreover the second half of condition (**S2**) follows from (V2), because (V2) implies that $V_0(\cdot,x)$ is continuous on $[0,T]$.

The following lemma guarantees that $S(t)$ satisfies conditon (**S3**):

Lemma 4.2. *Let $S(t)$ be as above. Then*

$$\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\| \le \left|\int_s^t \sigma_1(r)\,dr\right| \max_{r\in\{s,t\}} \|S(r)^{1/2}v\|, \quad v \in Y,\ t,s \in [0,T].$$

Proof. Since

$$\|S(t)^{1/2}v\|^2 = \|(H_0+V_0(t))v\|^2 + \||x|v\|^2 + \|v\|^2,$$

we have

$$\frac{d}{dt}\|S(t)^{1/2}v\|^2 = 2\Big(\frac{\partial}{\partial t}V_0(t)v, (H_0+V_0(t))v\Big).$$

Therefore we obtain from the Schwartz inequality and Hardy inequality that

$$\left|\frac{d}{dt}\|S(t)^{1/2}v\|^2\right| = 2\sigma(t)(\|2a\nabla v\| + \|v\|)\|(H_0+V_0(t))v\|.$$

It follows from (4.1) that

$$\left|\frac{d}{dt}\|S(t)^{1/2}v\|\right| = (2a+b)c_2\sigma(t)\|S(t)^{1/2}v\|.$$

Thus the assertion follows by $\sigma_1(t) = (2a+b)c_2\sigma(t)$. $\square$

Now we shall verify conditions (**A1**)–(**A4**).

Lemma 4.3. *Let* $A(t) = iH(t)$ *and* $S(t)$ *be as above. Then for each* $T > 0$

(A1) $\mathrm{Re}\,(A(t)v, v) = 0$, $v \in D(A(t))$, a.a. $t \in (0, T)$.

(A2) $Y = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4 \subset D(A(t))$, a.a. $t \in (0, T)$.

(A3) *There exists a constant* $\beta \geq 0$ *such that*

$$|\,\mathrm{Re}\,(A(t)u, S(t)u)| \leq \beta \|S(t)^{1/2}u\|^2, \ u \in D(S(t)), \ \text{a.a. } t \in (0, T).$$

(A4) $A(\cdot) \in C_*(0, T; B(\Sigma^1(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4))$.

Proof. Noting that $\mathrm{Re}\,(A(t)u, u) = -\,\mathrm{Im}\,(H(t)u, u)$ the assertion follows from symmetry of $H(t)$. Moreover the continuity of $A(t)$ follows from (V2) and (q). Therefore, it is sufficient to show that there exist $\beta > 0$ such that

$$|\,\mathrm{Im}\,(H(t)u, S(t)u)| \leq \beta(t) \|S(t)^{1/2}u\|^2, \quad u \in D(S), \ \text{a.a. } t \in (0, T).$$

By integration by parts we have

$$\begin{aligned} \mathrm{Im}\ (H(t), S(t)u) &= \mathrm{Im}\,\big((H_0 + V_0(t))u, |x|^2u\big) + \mathrm{Im}\,\big(q(t)u, (H_0 + V_0(t))^2u\big) \\ &= \mathrm{Re}\,\big((\alpha \cdot x)u, u\big) - \mathrm{Re}\,\big((\alpha \cdot \nabla q(t))u, (H_0 + V_0(t))u\big). \end{aligned}$$

We see from condition (q) that there exists a constant $c_q > 0$ such that

$$\|(1 + |x|^2)^{-1/2}\nabla q(t)\|_{L^\infty} \leq c_q.$$

Hence it follows from the Schwarz inequality and that

$$\begin{aligned} \big|\mathrm{Im}\,(H(t), S(t)u)\big| &\leq \||x|u\| \cdot \|u\| + \||\nabla q(t)|u\| \cdot \|(H_0 + V_0(t))u\| \\ &\leq \||x|u\| \cdot \|u\| + c_q\|(1 + |x|^2)^{1/2}u\| \cdot \|(H_0 + V_0(t))u\|, \end{aligned}$$

Therefore we obtain the desired inequality. □

According to the above lemmas, we can obtain Theorem 1.1.

References

[1] T. Kato, "Perturbation Theory for Linear Operators", Grundlehren der mathematischen Wissenschaften **132**, Springer-Verlag, Berlin and New York, 1966; 2nd ed., 1976.

[2] T. Kato, *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Univ. Tokyo, Sec. I. **17** (1970), 241–258.

[3] T. Kato, *Singular perturbation and semigroup theory*, in Lecture Notes in Math. **565**, Springer-Verlag, Berlin and New York, 1976, 104–112.

[4] T. Kato, *Abstract Differential Equations and Nonlinear Mixed Problems*, Lezioni Fermiane [Fermi Lectures], Accad. Naz. Lincei, Scuola Normale Superiore, Pisa, 1985.

[5] T. Kato and K. Yajima, *Dirac equations with moving nuclei*, Ann. Inst. H. Ponicaré, Phys. Théor. **54** (1991), 209–221.

[6] N. Okazawa, *Remarks on linear m-accretive operators in a Hilbert space*, J. Math. Soc. Japan **27** (1975), 160–165.

[7] N. Okazawa, *Remarks on linear evolution equations of hyperbolic type in Hilbert space*, Adv. Math. Sci. Appl. **8** (1998), 399–423.

[8] N. Okazawa and K. Yoshii, *Abstract approach to the Dirac equation*, RIMS Kôkyûroku **1640** (2009), 67–84.

[9] N. Okazawa and K. Yoshii, *Linear evolution equations with strongly measurable families and application to the Dirac equation*, Discrete Contin. Dyn. Syst. Series S **4** (2011), 723–744.

[10] N. Okazawa and K. Yoshii, *Linear Schrödinger evolution equations with moving Coulomb singularities*, J. Differential Equations **254** (2013), 2964–2999.

[11] E. M. Ouhabaz, "Analysis of Heat Equations on Domains", London Math. Soc. Monograph, Princeton Univ. Press, Princeton and Oxford, 2005.

[12] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Applied Math. Sci. **44**, Springer-Verlag, Berlin and New York, 1983.

[13] H. Tanabe, "Functional Analytic Methods for Partial Differential Equations", Pure and Applied Mathematics **204**, Marcel Dekker, New York, 1997.